The solutions of the diffusion-type equations, [Eqs (6c) and (6d)], subject to the boundary conditions, [Eq (7)], are

$$U^2(x)\bar{u}(x,y) = \frac{U_e^2(0)}{2x^{(1+m)/2}} \int_\alpha^\infty \xi^m \bar{u}_0(\xi) G(\xi; x,y) d\xi - \frac{1}{4}\int_0^x \frac{dU_e^2}{d\tau} \frac{d\tau}{\{x + \tau[(1-\sigma)/\sigma]\}^{(m+1)/2}} \times \int_\alpha^\infty \xi^m \bar{h}_0(\xi) G\left(\xi; x + \tau\frac{(1-\sigma)}{\sigma}, y\right) d\xi \quad (8a)$$

$$\bar{h}(x,y) = \frac{1}{2}\left(\frac{\sigma}{x}\right)^{(1+m)/2} \int_\alpha^\infty \xi^m \bar{h}_0(\xi) G\left(\xi; \frac{x}{\sigma}, y\right) d\xi \quad (8b)$$

where

$$\alpha = -\infty \qquad m = 0$$

$$\alpha = 0 \qquad m = 1$$

$$G(\xi; x,y) = (1/\pi^{1/2}) \exp -(y-\xi)^2/4x \qquad m = 0$$

$$= I_0(y\xi/2x) \exp -(y^2 + \xi^2)/4x \qquad m = 1 \quad (9)$$

The momentum thickness θ^*, and net heat transfer to the body Q^*, are given by

$$\theta^{*m+1}(x^*) = 2^m \int_\alpha^\infty \frac{\rho^* u^*}{\rho^* U^*}\left(1 - \frac{u^*}{U^*}\right) y^{*m} dy^* \quad (10a)$$

$$Q^* = 2^m \pi^m \int_\alpha^\infty \rho^* u^* (H^* - H^*) y^{*m} dy^* \quad (10b)$$

and to first order by

$$\Theta^{m+1}(x) = \frac{\rho_e^* U^* \theta^{*m+1}}{\rho_\infty^* U_\infty^* d^{*m+1}} \left(\frac{RC}{\pi}\right)^{(m+1)/2} \left(\frac{\pi^m}{2^{1+m}}\right) = 2^{-1}\pi^{(m-1)/2} \int_\alpha^\infty \bar{u}(x,y) y^m dy$$

$$= \frac{U^2(0)}{U^2(x)}\Theta^{m+1}(0) - \frac{1}{2}\left[1 - \frac{U_e^2(0)}{U^2(x)}\right] \times [(\gamma - 1) M^2(0)\Theta^{m+1}(0) - \Phi(0)] \quad (11a)$$

$$\Phi(x) = \frac{Q^*}{2^{1+m}\rho_\infty^* U_\infty^* d^{*m+1} h^*(0)} \left(\frac{RC}{\pi}\right)^{(m+1)/2}$$

$$= 2^{-1}\pi^{(m-1)/2} \int_\alpha^\infty [U^2(x)\bar{u}(x,y) - h(x)\bar{h}(x,y)] y^m dy = \Phi(0) \quad (11b)$$

where

$$\int_\alpha^\infty \xi^m \bar{u}_0(\xi) d\xi = 2\pi^{(1-m)/2}\Theta^{m+1}(0)$$

$$\int_\alpha^\infty \xi^m \bar{h}_0(\xi) d\xi = 2\pi^{(1-m)/2} [(\gamma - 1) M_e^2(0)\Theta^{m+1}(0) - \Phi(0)] \quad (12)$$

If $\bar{u}_0(y)$ and $\bar{h}_0(y)$ are taken to be delta functions, Eq (8) becomes

$$U^2(x)\bar{u}(x,y) = \frac{\Theta^{m+1}(0)}{x^{(m+1)/2}} U_e^2(0) e^{-y^2/4x} - \frac{1}{2}[(\gamma-1) M^2(0)\Theta^{m+1}(0) - \Phi(0)] \times \int_0^x \frac{dU_e^2}{d\tau} \frac{\exp(y^2/4\{x + \tau[(1-\sigma)/\sigma]\})}{\{x + \tau[(1-\sigma)/\sigma]\}^{(m+1)/2}} d\tau \quad (13a)$$

$$\bar{h}(x,y) = [(\gamma - 1) M^2(0)\Theta^{m+1}(0) - \Phi(0)] \times (\sigma/x)^{(m+1)/2} e^{-y^2\sigma/4x} \quad (13b)$$

Thus, although the enthalpy profile is still similar, the velocity profile is no longer similar when $dU^*/dx^* \neq 0$ It can be shown also [from Eq (8)] that for large values of the streamwise coordinate x, the velocity and enthalpy distributions are Gaussian to first order regardless of the initial distributions

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Viscous Jets from Nonnarrow Orifices

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The efflux of a viscous jet from an orifice has been studied in an exact way by means of boundary-layer theory by Schlichting,[1] when the orifice is a narrow slit and the surrounding fluid is at rest In this case, similar solutions do exist

If the outer medium is moving, nearly similar solutions have been given by Pozzi and Sabatini [2] For orifices of nonvanishing width and constant outer velocity, there is a linearized solution

The purpose of this note is to give a more general solution of this problem (wide orifices and surrounding medium not at rest), offering a method that takes into account the axial pressure gradient and whose second approximation can be obtained in closed form

Basic Equations

The equations governing a two-dimensional incompressible viscous flow in the boundary-layer theory in nondimensional form are as follows:

Momentum

$$uu_x + vu_y = U U_x + u_{yy} \quad (1)$$

Continuity

$$u_x + v_y = 0 \quad (2)$$

where $U_e(x)$ is the outer velocity

The boundary conditions for the efflux of jets are

$$u(x,\infty) = U$$

$$u(0,y) = \begin{cases} U(0) & |y| > h/2 \\ U_i & |y| < h/2 \end{cases}$$

$$v(x,0) = 0 \text{ (by symmetry)}$$

where h is the width of orifice and U_i is the initial velocity of the jet

The linearized solution of Ref 3 is obtained by putting $u = U_e + u_1$, $v = v_1$, and considering $U = \text{const} = 1$ Equation (1) thus becomes

$$u_{1x} = u_{1yy} \quad (3)$$

whose solution is

$$u_1 = \tfrac{1}{2}(U_i - U)\left[\frac{\text{erf}(h/2 + y)}{2x^{1/2}} + \frac{\text{erf}(h/2 - y)}{2x^{1/2}}\right] \quad (4)$$

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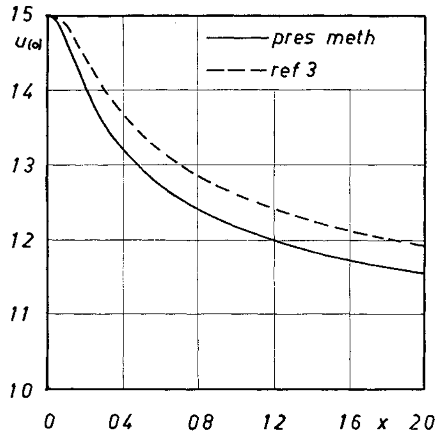

Fig 1 Comparison between u_1 of Ref 3 and the present second approximation

The writer doubts that this type of linearization is the most accurate or the most suitable for taking into account the influence of an axial pressure gradient

Indeed, in a field very close to that of jets, i e , the mixing of two streams, a different type of linearization was successfully used by Napolitano and Pozzi [4 5] In this case, a comparison with exact solutions was possible, and the accuracy of the method was found satisfactory

By means of this approach, the first and second approximations are obtained in closed form

Let us introduce, therefore, the nondimensional von Mises coordinates, thus defined:

$$\xi = \int_0^\xi U\,dx \tag{5}$$

$$\eta = \psi \tag{6}$$

where ψ is the nondimensional stream function Putting $Z = U^2 - u^2$, Eq (1) and boundary condition become

$$Z_1 = (1 - Z/U^2)^{1/2} Z_{\eta\eta} \tag{7}$$

$$Z(\xi, \infty) = 0 \tag{8}$$

$$Z(0,\eta) = \begin{cases} 0 & |\eta| > 1 \\ U_e^2(0) - U_i^2 & |\eta| < 1 \end{cases} \tag{9}$$

where the reference length is chosen in such a way that the height of the orifice in von Mises plane is equal to 2

If $[U_i - U(0)]/U(0) \ll 1$, it is possible to expand Z in series of a parameter, depending on the difference between the velocity of the two streams

In Ref 4 it has been found that the best parameter is $\frac{1}{2}(U^2 - U_i^2)$ Hence, the equations and boundary conditions for the first and second approximation are

$$Z_{1\xi} = Z_{1\eta\eta} \tag{10}$$

$$Z_{2\xi} = Z_{2\eta\eta} - (1/2U^2) Z_1 Z_{1\xi} \tag{11}$$

$$Z_1(\xi, \infty) = Z_2(\xi, \infty) = 0$$

$$Z_1(0,\eta) = \begin{cases} 0 & |\eta| > 1 \\ 2 & |\eta| < 1 \end{cases} \qquad Z_2(0, \eta) = 0$$

Solutions of the Equations

Equations (10) and (11) are of the heat-conduction type and can be solved by means of Green's function

One has, indeed,

$$Z_i = \int_0^\xi \int_{-\infty}^{\infty} R_i(x,y) G(x,y;\xi,\eta)\,dx\,dy + \int_{-\infty}^{\infty} Z_i(0,y) G(0,y;\xi,\eta)\,dy \tag{12}$$

if the equation governing Z_i is

$$Z_{i\xi} = Z_{i\eta\eta} + R_i$$

and $G(x,y;\xi,\eta)$ is the Green function for the half plane given by

$$G(x,y:\xi,\eta) = \tfrac{1}{2}[\pi(\xi - x)]^{-1/2} \exp[-(\eta - y)^2/4(\xi - x)]$$

The first approximation Z_1 yields

$$Z_1 = \operatorname{erf}\frac{1 + \eta}{2\xi^{1/2}} + \operatorname{erf}\frac{1 - \eta}{2\xi^{1/2}} \tag{13}$$

For Z_2, it is possible to obtain a general integral expression by integrating with respect to y One thus has

$$Z_2 = f_1 + f_2$$

with

$$f_i = \frac{\alpha_i i^{-1}(\alpha_i)}{4\xi} \int_0^\xi \left\{ \operatorname{erf}\left[\frac{\xi^{1/2} - \alpha\ x}{x^{1/2}(2\xi - x)^{1/2}}\right] + \operatorname{erf}\frac{\alpha_i x^{1/2}}{(2\xi - x)^{1/2}} \right\} \times \frac{dx}{U^2} + \frac{\pi^{-1/2} i^{-1}(\alpha_i)}{4\xi} \int_0^\xi \frac{(x - \xi)}{(2\xi - x)^{1/2}} \left\{ \exp\left[\frac{\alpha_i x - \xi^{1/2}}{(2\xi - x)^{1/2} x^{1/2}}\right] - \exp\frac{\alpha_i^2 x}{x - 2\xi} \right\} \frac{dx}{U^2} \tag{14}$$

where i^{-1} is the derivate of erf, $\alpha_1 = (1 + \eta)/2\xi^{1/2}$, and $\alpha_2 = (1 - \eta)/2\xi^{1/2}$

If U^{-2} can be expanded in a power series, the integrals in the right-hand side of Eq (14) can be evaluated in a closed form

Now the case of constant U will be considered One has thus

$$4Z_2 = \tfrac{1}{2}\{[i^{-1}(\alpha_1)]^2 + [i^{-1}(\alpha_2)]^2\} + \alpha_1 i^{-1}(\alpha_1) \operatorname{erf}\alpha_1 + (+\alpha_2 i^{-1})(\alpha_2) \operatorname{erf}\alpha_2 - \alpha_1 i^{-1}(\alpha_2) \operatorname{erf}\alpha_1 + (-\alpha_2 i^{-})^1(\alpha_1) \operatorname{erf}\alpha_2 + (\alpha_1 + \alpha_2)[i^{-1}(\alpha_1) + (+i^{-1})(\alpha^2)] - i^{-1}(\alpha_1) i^{-1}(\alpha_2)$$

(For the evaluation of the integral, see the Appendix) Hence, one has for the velocity

$$u_1 = (1 - BZ_1)^{1/2} \qquad u_2 = (1 - BZ_1 - B^2 Z_2)^{1/2} \tag{15}$$

as first and second approximations, respectively

We note that u_1 is not a linear function of B; hence, a comparison with Eq (4) cannot be made without specifying the value of B At $y = \eta = 0$, Eqs (15) become

$$u_1 = (1 - 2B \operatorname{erf} 0\,5\xi^{-1/2})^{1/2}$$

$$u_2 = [1 - 2B \operatorname{erf} 0\,5\xi^{-1/2} + (-B^2 0\,5\xi^{-1/2}) i^{-1}(0\,5\xi^{-1/2})]^{1/2} \tag{16}$$

The present result, u_2, from Eq (16) and that of Ref 3, Eq (4), are plotted in Fig 1 for $U(0) - U_i = -0\,5$ (i e , $B = -0{,}625$); $\xi = x$ in this case, as $U = 1$ As one can see, the present solution gives velocities lower than that of Ref 3

It is the author's opinion that Eq (16) is a better approximation, as in the similar case of Ref 4; the same approach agreed well with the exact solution

Appendix

Of the integrals used in this study, two were unusual enough to be noted here The first one, already considered in Ref 4, is

$$\int_{-\infty}^{\infty} \operatorname{erfc}(a + bz) \exp(-z^2)\,dz = \pi^{1/2} \operatorname{erfc} a(1 + b^2)^{-1/2}$$

and is obtained by differentiating the left-hand side with respect to b and integrating first with respect to z and then with respect to b

The second type is

$$\int_0^1 \frac{\exp[-at^2 - bt^{-2}]}{(1 + t^2)^n} \tag{A1}$$

To evaluate this integral, one starts from the case $n = 0$ considered in Ref 6; one thus has

$$\int_0^1 \exp[-at^2 - bt^{-2}]dt =$$

$$\frac{1}{4}\left(\frac{\pi}{a}\right)^{1/2} \{\exp 2(ab)^{1/2} \operatorname{erf}(a^{1/2} + b^{1/2}) +$$

$$\exp[-2(ab)^{1/2}] \operatorname{erf}(a^{1/2} - b^{1/2}) - \exp 2(ab)^{1/2} + \exp[-2(ab)^{1/2}]\}$$

Now, multiplying both sides of this equation by $\exp(-a)$ and integrating with respect to a, one has at the left-hand side the case $n = 1$ of Eq (A1) The integral at the right-hand side can be integrated by parts

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Unsteady Boundary-Layer Flow of a Non-Newtonian Fluid on a Flat Plate

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Nomenclature

a = constant of proportionality defined by Eq (1)
C_f = coefficient of friction
f = transformed stream function
L = characteristic length
n = non Newtonian fluid index defined by Eq (1)
$\bar{U}$ = velocity at the edge of the boundary layer, divided by U_0
$\bar{u}$ = velocity in x direction, divided by U_0
$\bar{v}$ = velocity in y direction, divided by U_0
U_0 = reference velocity
$\bar{x}$ = coordinate along the surface, divided by L
$\bar{y}$ = coordinate normal to the surface divided by L
$\bar{t}$ = time, multiplied by U_0/L
R_{0n} = Reynolds number, $\rho U_0^{2-n}L^n/a$
α_1, α_2 = constants
τ = shear stress
τ_0 = shear stress at the wall
ρ = density
η = similarity variable
ξ = function of x and t

IN a recent report[1] in which an attempt was made to determine all possible similar solutions to the two-dimensional laminar boundary-layer equations for power-law non-Newtonian fluids, it was found that one of the unsteady flow solutions corresponds to the physical situation of impulsively started flow on a flat plate at zero angle of attack It is the purpose of this note to trace the development, by means of an interesting technique, of this particular solution from the general boundary-layer equations and similarity considerations In addition, numerical calculations are given for the case of impulsively started flow on an infinite flat plate, a simplification of the general problem

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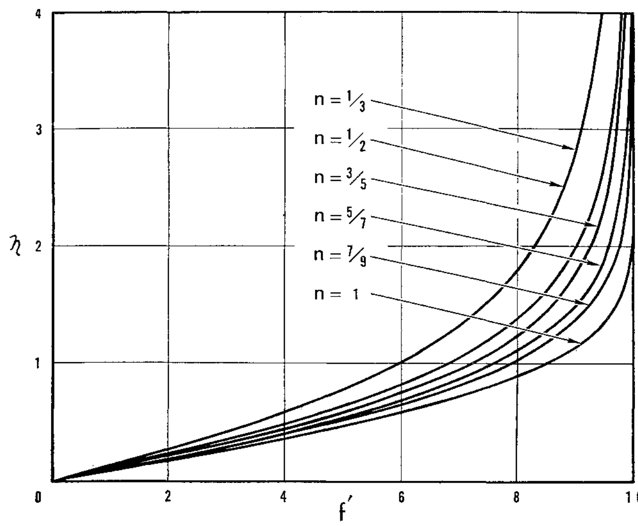

Fig 1 Similar boundary-layer profiles for impulsively started flow over an infinite flat plate for several non-Newtonian flow indices

The Ostwald-deWaele (power-law) model was chosen to describe the non-Newtonian fluid properties This is written, for the case of two-dimensional shear in the x direction due to a velocity gradient in the y direction, as

$$\tau_{yx} = a(du/dy)^n \tag{1}$$

where a and n are constant properties of the fluid

The model described by Eq (1) is purely phenomenological; however, it is useful in that it approximately describes a great number of real non-Newtonian fluids It can be shown, however, that this model behaves properly under tensor transformation Use of this model alone assumes that the fluid is purely viscous, that is, no elasticity effects or anisotropic normal stresses are present Values of n less than one describe fluids which are shear-thinning, and values of n greater than one describe fluids which are sheaı-thickening (known less descriptively as pseudoplastic and dilatant, respectively) For n equal to one, the expression describes Newtonian fluids, and a equals the coefficient of viscosity

The purpose of the following description of the analysis contained in Ref 1 is to outline the method used in obtaining the ordinary differential equation, the boundary conditions, and the similarity parameter that corresponds to the physical situation of unsteady flow near a flat plate when the inviscid freestream is suddenly put into motion The technique is that of treating what Fenter terms the inverse problem—manipulating the general boundary-layer equations to find the conditions under which mathematical similarity (velocity profiles which transform linearly in y) exists [2]

The incompressible two-dimensional boundary-layer equations for power-law non-Newtonian fluids can be combined and transformed into a single ordinary differential equation in terms of a stream function $f' = \bar{u}/\bar{U}$ and a single independent variable η η is in general a function of $\bar{y}$ and $\xi(\bar{x},\bar{t})$, the form of which is dictated by similarity and the form of the differential equation The differential equation contains coefficients that must be constants in order to satisfy similarity With the assumption that the inviscid flow velocity is